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LITTLEWOOD-PALEY AND MULTIPLIER THEOREMS ON WEIGHTED L^p SPACES[1]

BY

DOUGLAS S. KURTZ

ABSTRACT. The Littlewood-Paley operator $\gamma(f)$, for functions f defined on $\mathbf{R}^n$, is shown to be a bounded operator on certain weighted L^p spaces. The weights satisfy an A_p condition over the class of all n-dimensional rectangles with sides parallel to the coordinate axes. The necessity of this class of weights demonstrates the 1-dimensional nature of the operator. Results for multipliers are derived, including weighted versions of the Marcinkiewicz Multiplier Theorem and Hörmander's Multiplier Theorem.

1. Introduction. Let $m(x)$ be a bounded functon on $\mathbf{R}^n$. The operator Tf defined by the Fourier transform equation $(Tf)\hat{}(x) = m(x)\hat{f}(x)$ is called a multiplier operator with multiplier $m(x)$. Let ρ be an (n-dimensional) interval and $\chi_\rho(x)$ the characteristic functon of ρ. The operator $S_\rho f$, having multiplier $m(x) = \chi_\rho(x)$ and defined by the equation

$$(S_\rho f)\hat{}(x) = \chi_\rho(x)\hat{f}(x),$$

is called a partial sum operator.

We define the operator $\gamma(f)$ by

DEFINITION 1.1. Let a collection of disjoint intervals $\Delta = \{\rho\}$ be a decomposition of $\mathbf{R}^n$ (i.e., $\bigcup_\Delta \rho = \mathbf{R}^n$). Given a function, f, in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$, define

$$\gamma(f)(x) = \gamma(f, \Delta)(x) = \left(\sum_\Delta |S_\rho f(x)|^2\right)^{1/2}. \tag{1.1}$$

By taking Fourier transforms, for any decomposition Δ, we have the obvious L^2 equality

$$\|\gamma(f)\|_2 = \|f\|_2. \tag{1.2}$$

Given an appropriate Δ, we will show that (1.2) can be extended to certain weighted L^p spaces as an equivalence between norms.

A sequence $\{n_k\}_{k=-\infty}^{+\infty}$, $n_k > 0$, is called a lacunary sequence if there is an $\alpha > 1$ such that $n_{k+1}/n_k \geq \alpha$ for all k. The dyadic sequence, $n_k = 2^k$, is an example of such a sequence.

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DEFINITION 1.2. Let $\{n_k\}_{k=-\infty}^{+\infty}$ be a lacunary sequence. Let Δ be the collection of all intervals of the form $[n_k, n_{k+1}]$ and $[-n_{k+1}, -n_k]$, $-\infty < k < \infty$. Then, Δ is called a lacunary decomposition of $\mathbf{R}^1$.

It follows from the definition of a lacunary sequence that $\bigcup_\Delta \rho = \mathbf{R}^1$ (or, to be exact, $\mathbf{R}^1 - \{0\}$). When $\{n_k\}$ is the dyadic sequence, the resulting Δ is called the dyadic decomposition of $\mathbf{R}^1$.

DEFINITION 1.3. Let Δ_i, $i = 1, 2, \ldots, n$, be n lacunary decompositions of $\mathbf{R}^1$. Let Δ be the collection of the intervals, ρ, of the form $\rho = \rho_1 \times \rho_2 \times \cdots \times \rho_n$ where $\rho_i \in \Delta_i$. Then, Δ is called a lacunary decomposition of $\mathbf{R}^n$.

It is well known (see **[21]** and **[24]**) that if Δ is a lacunary decomposition of $\mathbf{R}^n$, then $\|\gamma(f)\|_p$ is equivalent to $\|f\|_p$ for $1 < p < \infty$; i.e., there are constants $A(p, \Delta)$ and $B(p, \Delta)$ such that

$$A(p, \Delta)\|f\|_p \leqslant \|\gamma(f)\|_p \leqslant B(p, \Delta)\|_p\|f\|_p. \tag{1.3}$$

The weight functions we will consider satisfy the following definition.

DEFINITION 1.4. Let $\mathfrak{R}$ be a collection of bounded sets in $\mathbf{R}^n$ and w a nonnegative, locally integrable function. If $1 < p < \infty$, then w is in $A_p(\mathbf{R}^n, \mathfrak{R})$ if there is a constant, c, such that

$$\left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant c$$

for any $R \in \mathfrak{R}$. We say w is in $A_1(\mathbf{R}^n, \mathfrak{R})$ if there is an constant, c, such that $w^*(x) \leqslant cw(x)$ for almost every x, where

$$w^*(x) = \sup_{\substack{R\in\mathfrak{R}\\ x\in R}} \frac{1}{|R|}\int_R w(x)\,dx$$

is the Hardy-Littlewood maximal function of w with respect to the collection $\mathfrak{R}$.

This class of functions was first introduced by Rosenblum **[17]** and Muckenhoupt **[11]**. The basic properties of A_p functions can be found in Muckenhoupt **[11]** and Coifman and C. Fefferman **[1]**.

Let $\mathfrak{Q}_n$ and $\mathfrak{R}_n$ denote the collections of all n-dimensional cubes and all n-dimensional intervals with sides parallel to the coordinate axes, respectively. $A_p(\mathbf{R}^n, \mathfrak{Q}_n)$ is the A_p class of Muckenhoupt. We note that when $n = 1$, $A_p(\mathbf{R}^1, \mathfrak{Q}_1) = A_p(\mathbf{R}^1, \mathfrak{R}_1)$. However, for $n > 1$, we have $A_p(\mathbf{R}^n, \mathfrak{R}_n) \subsetneq A_p(\mathbf{R}^n, \mathfrak{Q}_n)$. That the containment is strict is demonstrated by the fact that $|x|^\alpha \in A_p(\mathbf{R}^n, \mathfrak{Q}_n)$ for $-n < \alpha < n(p-1)$ while $|x|^\alpha \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$ for $-1 < \alpha < p-1$. In other words, the values of α for which $|x|^\alpha$ is in $A_p(\mathbf{R}^n, \mathfrak{R}^n)$ lie in the 1-dimensional range, $-1 < \alpha < p - 1$. As we will see in the next section, this is a consequence of the fact that $A_p(\mathbf{R}^n, \mathfrak{R}_n)$ can be described as the class of functions which are in $A_p(\mathbf{R}^1, \mathfrak{R}_1) = A_p(\mathbf{R}^1, \mathfrak{Q}_1)$ in each variable uniformly with respect to the other variables.

Let $w(x)$ be a nonnegative function. We define $L^p_w(\mathbf{R}^n)$, $1 \leqslant p < \infty$, to be the collection of all functions f such that $\int_{\mathbf{R}^n}|f(x)|^p w(x)\,dx < +\infty$. For $f \in L^p_w(\mathbf{R}^n)$, we define

$$\|f\|_{p,w} = \left(\int_{\mathbf{R}^n}|f(x)|^p w(x)\,dx\right)^{1/p}.$$

$\| \|_{p,w}$ is a norm which makes $L^p_w(\mathbf{R}^n)$ a Banach space. Closely allied to the L^p_w spaces are the weighted analogs of the Hardy-Stein-Weiss spaces H^p. We assume the reader is familiar with the theory of H^p spaces and their relationships to L^p spaces. For $p > 1$ and $w \in A_p(\mathbf{R}^n, \mathcal{Q}_n)$, H^p_w is naturally isomorphic to L^p_w. When $p = 1$ and $w \in A_1(\mathbf{R}^n, \mathcal{Q}_n)$, H^1_w is isomorphic to a subspace of L^1_w. For definitions and details, see [3], [12], [21], [22], and [24]. The Schwartz class, $\mathcal{S}(\mathbf{R}^n)$, of infinitely differentiable functions of rapid decrease at infinity is dense in all of the previously mentioned spaces, in the appropriate norms.

We now state our main result.

THEOREM 1. *Let $\Delta = \{\rho\}$ be a lacunary decomposition of $\mathbf{R}^n$, $1 < p < \infty$, and $w \in A_p(\mathbf{R}^n, \mathcal{R}_n)$. Then, there are constants A and B, depending on p, w, and Δ, such that*

$$A\|f\|_{p,w} \leqslant \|\gamma(f)\|_{p,w} \leqslant B\|f\|_{p,w} \tag{1.4}$$

When Δ is the dyadic decomposition of $\mathbf{R}^n$, (1.4) implies $w \in A_p(\mathbf{R}^n, \mathcal{R}_n)$.

Hirschman [4] proved this theorem in the periodic case, where Δ is the dyadic decomposition of the integers and $w(\theta) = |\theta|^a$, $-1 < a < p - 1$. When $n = 1$ and Δ is the dyadic decomposition of $\mathbf{R}^1$, the theorem extends to a weak type result for $p = 1$. That is, there exists a constant, c, depending on w and Δ, such that

$$w(\{x \in \mathbf{R}^n: |\gamma(f)(x)| > \lambda\}) \leqslant (c/\lambda)\|f\|_{H^1_w},$$

where, given a measurable set E, $w(E) = \int_E w(x)\, dx$.

The proof of Theorem 1 is divided into several parts. We first show that in $\mathbf{R}^1$, $w \in A_p$ and $f \in L^p_w$ imply (1.4). From this we derive the n-dimensional version. Next we show that (1.4) implies $w \in A_p$ when Δ is the dyadic decomposition of $\mathbf{R}^n$. The proof is completed by showing that if $w \in A_p$ and $f \notin L^p_w$ then $\gamma(f) \notin L^p_w$.

In order to go from the dyadic decomposition of $\mathbf{R}^1$ to a general lacunary one, we need a generalization of the Marcinkiewicz Multiplier Theorem.

THEOREM 2. *Let m be bounded on $\mathbf{R}^1$ and of bounded variation on every finite interval not containing the origin. Let $\|m\|_\infty \leqslant B$ and $\int_I |dm(x)| \leqslant B$ for every dyadic interval I. If $1 < p < \infty$ and $w \in A_p(\mathbf{R}^1, \mathcal{R}_1)$, then m is a bounded multiplier from $L^p_w(\mathbf{R}^1)$ to $L^p_w(\mathbf{R}^1)$, with norm depending only on B, p, and w.*

As in the unweighted case, Theorem 1 is equivalent to Theorem 2 when $n = 1$. Using the result for the γ-function in $\mathbf{R}^n$, we can get a generalization of Theorem 2. We think of $\mathbf{R}^n$ as divided into 2^n "quadrants" by the coordinate axes. For example, the first "quadrant" is the set $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n: x_i > 0, i = 1, \ldots, n\}$.

THEOREM 3. *Let $m \in C^n$ in each "quadrant" of $\mathbf{R}^n$ and such that $\|m\|_\infty \leqslant B$,*

$$\sup_{x_{k+1}, \ldots, x_n} \int_\rho \left| \frac{\partial^k m(x)}{\partial x_1 \cdots \partial x_k} \right| dx_1 \cdots dx_k \leqslant B$$

for $0 < k \leqslant n$, ρ any dyadic interval in $\mathbf{R}^k$, and any permutation of $(x_1, \ldots, x_n)$. If $1 < p < \infty$ and $w \in A_p(\mathbf{R}^n, \mathcal{R}_n)$ then m is a bounded multiplier from $L^p_w(\mathbf{R}^n)$ to $L^p_w(\mathbf{R}^n)$.

The proof of Theorem 1 relies on a weighted version of Hörmander's Multiplier Theorem.

THEOREM 4. *Let $k > [n/2]$ and $m \in C^k(\mathbf{R}^n - \{0\})$. Suppose that*

$$\sup_{r>0} r^{2|\alpha|-n} \int_{r<|x|<2r} \left| \left(\frac{\partial}{\partial x} \right)^{\alpha} m(x) \right|^2 dx \leq B, \quad \text{for } |\alpha| \leq k.$$

When $k < n$ and $n/k < p < \infty$, m is a bounded multiplier from $L_w^p(\mathbf{R}^n)$ to $L_w^p(\mathbf{R}^n)$ and maps $L_w^{n/k}(\mathbf{R}^n)$ to weak $L_w^{n/k}(\mathbf{R}^n)$ if $w \in A_{pk/n}(\mathbf{R}^n, \mathcal{Q}_n)$. For $k = n$, the strong type result is the same, but now $H_w^1(\mathbf{R}^n)$ gets mapped into weak $L_w^1(\mathbf{R}^n)$. Finally, if $k > n$, m is a bounded multiplier from $L_w^p(\mathbf{R}^n)$ to $L_w^p(\mathbf{R}^n)$, $1 < p < \infty$, and from $H_w^1(\mathbf{R}^n)$ to $H_w^1(\mathbf{R}^n)$. The norm of the operator depends only on B, p, w, k, and n.

We will prove Theorem 4 using Littlewood-Paley theory. A more general version of the theorem appears in **[10]**. That proof involves the sharp function of Fefferman and Stein **[2]**.

In Chapter 2, we prove several preliminary results, including Theorem 4. Chapter 3 consists of the proof of Theorem 1.

Chapter 4 is devoted to obtaining multiplier theorems. As a consequence, we obtain the following weighted, n-dimensional variant of the Hausdorff-Young Theorem.

$$\left(\sum_{\Delta} \|(S_\rho f)\hat{}(x)|x|^{-\alpha}\|_{p'}^2 \right)^{1/2} \leq C \|f(x)|x|^{\alpha}\|_p,$$

for $1 < p \leq 2$ and $0 \leq \alpha < 1/p'$ (see (4.5)). Typical of the results (though easier to state) is the following one.

THEOREM 5. *Let $1 < p \leq 2 \leq q < \infty$, $1/r = 1/p - 1/q$, $0 \leq \alpha < 1/q$, and $0 \leq \beta < 1/p'$. Given a bounded function $m(x)$, let Tf be the multiplier operator defined by $(Tf)\hat{}(x) = m(x)\hat{f}(x)$. If $m(x)|x|^{(\alpha+\beta)n} \in L^r(\mathbf{R}^n)$, then T is a bounded operator from $L^p_{|x|^{\beta np}}(\mathbf{R}^n)$ to $L^q_{|x|^{-\alpha nq}}(\mathbf{R}^n)$.*

We note that the main theorem and the applications are all true when carried out in the periodic case.

Throughout this paper, C will denote a positive constant, not necessarily the same for each occurrence, depending only on the parameters mentioned or implied but not on f (except in the proof of Theorem 4.2). All sets and functions mentioned are assumed to be measurable and we take $0 \cdot \infty$ to be 0.

I would like to express my warmest thanks to Dr. Richard L. Wheeden. This work is truly a product of his patience and guidance.

2. Preliminary results. Let $f = \{f_k\}$ be a vector-valued function on $\mathbf{R}^n$. We say $f \in L_w^p(\mathbf{R}^n, l^2)$ if $|f| = (\sum_k |f_k|^2)^{1/2} \in L_w^p(\mathbf{R}^n)$, and then $\|f\|_{L_w^p(\mathbf{R}^n, l^2)} = \|(\sum |f_k|^2)^{1/2}\|_{p,w}$. In addition to Theorem 4, we will obtain the following

THEOREM 2.1. *Let* $1 < p < \infty$ *and* $w \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$. *Let* $\Delta = \{\rho_k\}$ *be any collection of intervals in* $\mathbf{R}^n$. *For* $f = \{f_k\} \in L^p_w(\mathbf{R}^n, l^2)$, *set* $S(f) = \{S_{\rho_k} f_k\}$. *Then*

(i) $\|S(f)\|_{L^p_w(\mathbf{R}^n, l^2)} \leq C\|f\|_{L^p_w(\mathbf{R}^n, l^2)}$, $1 < p < \infty$,

(ii) $w(\{x \in \mathbf{R}^n: |S(f)(x)| > \lambda\}) \leq (C/\lambda)\|f\|_{L^1_w(\mathbf{R}^n, l^2)}$.

The C depends on w, p, *and* n.

LEMMA 2.2. *Let* $w \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$, $1 < p < \infty$. *Then, there exists a constant, C, depending only on w, such that for almost every fixed* $(n-1)$*-tuple,* $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$, *and any interval* $I \subset \mathbf{R}^1$,

$$\left(\frac{1}{|I|}\int_I w(x_1, \ldots, x_j, \ldots, x_n)\,dx_j\right) \cdot \left(\frac{1}{|I|}\int_I \{w(x_1, \ldots, x_j, \ldots, x_n)\}^{-1/(p-1)}\,dx_j\right)^{p-1} \leq C.$$

Lemma 2.2 says that A_p over arbitrary rectangles implies A_p in each variable uniformly with respect to the other variables. The two conditions are actually equivalent.

Let $f(x) \in \mathcal{S}(\mathbf{R}^n)$ and let $f(x, y)$ be its Poisson integral. Let $\nabla f(x, y)$ be the full gradient of $f(x, y)$ and define the kth gradient of f by

$$\nabla^k f(x, y) = \left(\nabla^{k-1}\frac{\partial}{\partial x_1} f(x, y), \ldots, \nabla^{k-1}\frac{\partial}{\partial x_n} f(x, y), \nabla^{k-1}\frac{\partial}{\partial y} f(x, y)\right).$$

In the proof of Theorem 4, we will need the following variants of the Littlewood-Paley g-function of f:

$$S_k(f)(x) = \left(\int\int_{|x-t|<y} |\nabla^k f(t, y)|^2 y^{2k-1-n}\,dt\,dy\right)^{1/2},$$

$$g^*_\lambda(f)(x) = \left(\int_0^\infty \int_{R_n}\left(\frac{y}{|t| + y}\right)^{\lambda n} |\nabla f(x - t, y)|^2 y^{1-n}\,dt\,dy\right)^{1/2}.$$

$S_k(f)$ satisfies the inequality

$$S_1(f)(x) \leq C_k S_k(f)(x) \tag{2.1}$$

(see [**21**, p. 216]).

We now begin the proof of Theorem 4. Let $f \in \mathcal{S}(\mathbf{R}^n)$ and define $g(x)$ by $\hat{g}(x) = m(x)\hat{f}(x)$. By standard arguments (see e.g., [**21**, pp. 96–99 and pp. 232–235]), we deduce that $S_{k+1}(g)(x) \leq Cg^*_\lambda(f)(x)$, $\lambda = 2k/n$. Therefore, by (2.1), $S_1(g)(x) \leq Cg^*_\lambda(f)(x)$. Applying Corollary 1 of [**3**] and the corollary to Theorem 2 of [**12**], we get $\|g\|_{H^p_w} \leq C\|f\|_{H^p_w}$ if $p > n/k$, or $p \geq 1$ if $k > n$. The result extends to H^p_w by continuity; for $p = 1$, see [**13**].

To prove the weak type inequalities, we need the nontangential maximal function of f, defined by

$$N(f)(x) = \sup_{\{(t,y):\,|x-t|<y/2\}} |f(t, y)|.$$

Since $f \in \mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n)$, $f(x) \leq N(f)(x)$ for almost every x. It follows from Gundy and Wheeden [**3**] that, given $0 < \varepsilon < 1$ and $\beta > 1$, there exists a $\delta > 0$

such that, for all $\alpha > 0$,

$$w(\{x \in \mathbf{R}^n: N(f)(x) > \beta\alpha\}) \leq \varepsilon w(\{x \in \mathbf{R}^n: N(f)(x) > \alpha\}) + w(\{x \in \mathbf{R}^n: S_1(f)(x) > \delta\alpha\}). \tag{2.2}$$

Multiply both sides of (2.2) by $(\beta\alpha)^{n/k}$ and take the sup over $\alpha > 0$. We then get

$$\sup_{\alpha>0} (\beta\alpha)^{n/k} w(\{x \in \mathbf{R}^n: N(f)(x) > \beta\alpha\}) \leq \varepsilon\beta^{n/k} \sup_{\alpha>0} \alpha^{n/k} w(\{x \in \mathbf{R}^n: N(f)(x) > \alpha\}) + \sup_{\alpha>0} (\beta\alpha)^{n/k} w(\{x \in \mathbf{R}^n: S_1(f)(x) > \delta\alpha\}).$$

Since the first sup on the right is the same as the one on the left, if we choose ε and β so that $\varepsilon\beta^{n/k} = \frac{1}{2}$ and change $\beta\alpha$ to α, we get

$$\sup_{\alpha>0} \alpha^{n/k} w(\{x \in \mathbf{R}^n: N(f)(x) > \alpha\}) \leq C \sup_{\alpha>0} \alpha^{n/k} w\left(\left\{x \in \mathbf{R}^n: S_1(f)(x) > \frac{\delta\alpha}{\beta}\right\}\right).$$

From the previous remarks and the corollary of [12], since $n/k = 2/\lambda$, and $g \in L^2(\mathbf{R}^n)$,

$$\begin{aligned}\sup_{\alpha>0} \alpha^{n/k} w(\{x \in \mathbf{R}^n: |g(x)| > \alpha\}) &\leq \sup_{\alpha>0} \alpha^{n/k} w(\{x \in \mathbf{R}^n: N(g)(x) > \alpha\}) \\ &\leq C \sup_{\alpha>0} \alpha^{n/k} w\left(\left\{x \in \mathbf{R}^n: S_1(g)(x) > \frac{\delta}{\beta}\alpha\right\}\right) \\ &\leq C \sup_{\alpha>0} \alpha^{n/k} w\left(\left\{x \in \mathbf{R}^n: g_\lambda^*(f)(x) > C'\frac{\delta}{\beta}\alpha\right\}\right) \\ &\leq C\|f\|_{H_w^{n/k}}^{n/k}.\end{aligned}$$

It is of interest to compare Theorem 4 to the original version of Hörmander's theorem. In the unweighted case, $k > [n/2]$ is sufficient to get a multiplier on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. This uses the fact that multipliers on L^p and $L^{p'}$ are the same. Although there is a similar duality for weighted L^p spaces, we cannot use it because we also need $w \in A_{pk/n}(\mathbf{R}^n, \mathcal{Q}_n)$. When $n = 1$ and $n = 2$, the two theorems agree since $k > [n/2]$ implies $k \geq n$ in these cases. Also, to get the H^1 result, both need the same k.

Let $\rho = (a, b) \subset \mathbf{R}^1$ and let $H(f)$ denote the Hilbert transform of f. By comparing the Fourier transforms, it is not hard to see that

$$S_\rho f(x) = S_{(a,b)} f(x) = \frac{i}{2}\left[e^{2\pi i x\cdot b} H(e^{-2\pi i t\cdot b} f(t))(x) - e^{2\pi i x\cdot a} H(e^{-2\pi i t\cdot a} f(t))(x)\right]. \tag{2.3}$$

Consider now Theorem 2.1. By (2.3), for the proof when $n = 1$, it is enough to know the result when $S(f)$ is replaced by the vector-valued Hilbert transform. This result was proved by John [9].

Next, let $\rho = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbf{R}^n$ and $H_j(f)$ be the 1-dimensional Hilbert transform in the x_j-variable. If S_ρ^j is the operator acting only on the x_j variable, by considering functions of the form $f(x) = f_1(x_1)f_2(x_2) \cdots f_n(x_n)$, it follows that $S_\rho f = S_\rho^n S_\rho^{n-1} \cdots S_\rho^1(f)$. But, as in (2.3),

$$S_\rho^j f(x) = \frac{i}{2}\left[e^{2\pi i x\cdot b_j} H_j(e^{-2\pi i t\cdot b_j} f(t))(x) - e^{2\pi i x\cdot a_j} H_j(e^{-2\pi i t\cdot a_j} f(t))(x)\right].$$

Thus the theorem is proved by an n-fold application of the 1-dimensional result once we establish Lemma 2.2; i.e., once we know that $w \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$ implies w is in A_p uniformly in each variable.

For simplicity in the proof of the lemma, we will assume $n = 2$. Let $w \in A_p(\mathbf{R}^2, \mathfrak{R}_2)$ and $I \subset \mathbf{R}^1$ be any interval. We want to show

$$\left(\frac{1}{|I|}\int_I w(x,y)\,dy\right)\left(\frac{1}{|I|}\int_I w(x,y)^{-1/(p-1)}\,dy\right)^{p-1} < C \tag{2.4}$$

for almost every x. If $J \subset \mathbf{R}^1$ is any interval, then by assumption

$$\left[\frac{1}{|J|}\int_J\left(\frac{1}{|I|}\int_I w(t,y)\,dy\right)dt\right]\left[\frac{1}{|J|}\int_J\left(\frac{1}{|I|}\int_I w(t,y)^{-1/(p-1)}\,dy\right)dt\right]^{p-1}$$
$$= \left(\frac{1}{|J\times I|}\int_{J\times I} w(t,y)\,dt\,dy\right)\left(\frac{1}{|J\times I|}\int_{J\times I} w(t,y)^{-1/(p-1)}\,dt\,dy\right)^{p-1} < C.$$

Letting J shrink to x and using Lebesgue's Differentiation Theorem, we get (2.4) for almost every x, depending on I. Considering only intervals, I, with rational endpoints and taking limits, the result follows.

3. Proof of Theorem 1. The proof of our main theorem will use Khinchine's inequality for Rademacher series. Let $r_m(t) = \operatorname{sgn}(\sin 2^m\pi t)$, $m = 0, 1, 2, \ldots,$ be the Rademacher functions, and set $f(t) = \sum_{m=0}^{\infty} a_m r_m(t)$. Then there are constants B_p and C_p such that for $0 < p < \infty$

$$B_p\left(\int_0^1 |f(t)|^p\,dt\right)^{1/p} < \left(\sum |a_m|^2\right)^{1/2} < C_p\left(\int_0^1 |f(t)|^p\,dt\right)^{1/p} \tag{3.1}$$

(see [**24**, Vol. I, p. 213]). If $m = (m_1, \ldots, m_n)$ is a multi-index of nonnegative integers, we define the n-dimensional Rademacher functions by

$$r_m(t) = r_m(t_1, \ldots, t_n) = r_{m_1}(t_1) r_{m_2}(t_2) \cdots r_{m_n}(t_n),$$

where $r_{m_i}(t_i)$ is a 1-dimensional Rademacher function. This collection satisfies an inequality similar to (3.1).

We begin the proof of Theorem 1 with

THEOREM 3.1. *Let Δ be the dyadic decomposition of* $\mathbf{R}^1$.

(i) *If* $1 < p < \infty$, $w \in A_p(\mathbf{R}^1, \mathfrak{R}_1)$, *and* $f \in L_w^p(\mathbf{R}^1)$, *then*

$$A\|f\|_{p,w} < \|\gamma(f)\|_{p,w} < B\|f\|_{p,w}.$$

(ii) *If* $w \in A_1(\mathbf{R}^1, \mathfrak{R}_1)$ *and* $f \in H_w^1(\mathbf{R}^1)$, *then*

$$w(\{x \in \mathbf{R}^1 : |\gamma(f)(x)| > \lambda\}) < (C/\lambda)\|f\|_{H_w^1}.$$

The constants A, B, and C depend only on p and w.

Let $\phi \in C^2(\mathbf{R}^1)$ be equal to 1 on (1, 2) and 0 on the complement of $(\frac{1}{2}, 4)$. For $\rho \in \Delta$, $\rho = [2^k, 2^{k+1}]$ say, set $\phi_\rho(x) = \phi(2^{-k}x)$ and define $\Phi_\rho f$ to be the operator with multiplier $\phi_\rho(x)$; i.e., $(\Phi_\rho f)\hat{}(x) = \phi_\rho(x)\hat{f}(x)$. Since $\phi_\rho(x) = 1$ for $x \in \rho$, it follows that

$$S_\rho \Phi_\rho f = S_\rho f. \tag{3.2}$$

Let $\{r_\rho(t)\}$ be the Rademacher functions indexed by $\rho \in \Delta$ and define

$$\psi_t f = \sum_\Delta r_\rho(t)\Phi_\rho f.$$

The multiplier associated with ψ_t is $m_t(x) = \sum_\Delta r_\rho(t)\phi_\rho(x)$. Since at most three ϕ_ρ's are nonzero for any given x, there is a constant B, depending only on ϕ, such that

$$|m_t(x)| \le B, \qquad \left|\frac{d}{dx} m_t(x)\right| \le \frac{B}{|x|}, \quad \text{and} \quad \left|\frac{d^2}{dx^2} m_t(x)\right| \le \frac{B}{|x|^2}.$$

Thus, $m_t(x)$ satisfies the conditions of Theorem 4, so that

$$\|\psi_t f\|_{p,w}^p \le C^p \|f\|_{p,w}^p \tag{3.3}$$

(if $p = 1$, these are H_w^p norms). Since B can be chosen independent of t, C does not depend on t. Now, integrate (3.3) in t from 0 to 1 and change the order of integration on the left. By (3.1),

$$\left\|\left(\sum_\Delta |\Phi_\rho f|^2\right)^{1/2}\right\|_{p,w} \le C\|f\|_{p,w}.$$

From Theorem 2.1 and (3.2), we get

$$\begin{aligned}\|\gamma(f)\|_{p,w} &= \left\|\left(\sum_\Delta |S_\rho f|^2\right)^{1/2}\right\|_{p,w} = \left\|\left(\sum_\Delta |S_\rho \Phi_\rho f|^2\right)^{1/2}\right\|_{p,w} \\ &\le C\left\|\left(\sum_\Delta |\Phi_\rho f|^2\right)^{1/2}\right\|_{p,w} \le C\|f\|_{p,w}\end{aligned} \tag{3.4}$$

for $p > 1$, and

$$\begin{aligned}(\{w \in \mathbf{R}^n: |\gamma(f)(x)| > \lambda\}) &= w\left(\left\{x \in \mathbf{R}^n: \left(\sum_\Delta |S_\rho \Phi_\rho f(x)|^2\right)^{1/2} > \lambda\right\}\right) \\ &\le (C/\lambda)\left\|\left(\sum_\Delta |\Phi_\rho f|^2\right)^{1/2}\right\|_{1,w} \le (C/\lambda)\|f\|_{H_w^1}.\end{aligned}$$

The proof of part (i) is completed by a duality argument (see **[21**, p. 105]).

Notice that Theorem 3.1 remains true if Δ is defined by the sequence $n_k = \alpha^k$, $\alpha > 1$, instead of $n_k = 2^k$. The proof is the same with trivial modifications.

In order to extend the result to a general lacunary decomposition of $\mathbf{R}^1$, we need Theorem 2. Assuming its validity for the moment, we will prove

THEOREM 3.2. *Let Δ be a lacunary decomposition of $\mathbf{R}^1$. If $1 < p < \infty$, $w \in A_p(\mathbf{R}^1, \mathfrak{R}_1)$, and $f \in L_w^p(\mathbf{R}^1)$, then there are constants A and B, depending only on p, w, and Δ, so that*

$$A\|f\|_{p,w} \le \|\gamma(f)\|_{p,w} \le B\|f\|_{p,w}.$$

Let Δ be defined by the sequence $\{\pm n_k\}_{k=-\infty}^{+\infty}$ with $n_{k+1}/n_k > \alpha > 1$. Let M be the least integer such that $\alpha^M > 2$. If $m(x)$ is any function such that $m|_\rho$ is identically $+1$ or -1 for each $\rho \in \Delta$, then $\|m\|_\infty = 1$ and $\int_I |dm(x)| < 2M$ for any dyadic interval I. By Theorem 2, m is a multiplier on $L_w^p(\mathbf{R}^1)$.

Define $m_t(x)$ by $m_t(x) = r_\rho(t)$ if $x \in \rho$. Then $m_t(x) = \Sigma_\Delta\, r_\rho(t)\chi_\rho(x)$. Also, let $G_N = \{x\colon\ 1/N < |x| < N\}$. $\Delta_N = \{\rho \in \Delta\ :\ \rho \subset G_N\}$ and define $m_t^N(x) = \Sigma_{\Delta_N} r_\rho(t)\chi_\rho(x)$. Clearly, m_t^N satisfies the same bounds as m_t. If we define g by $\hat{g}(x) = m_t^N(x)\hat{f}(x)$, then

$$\hat{g}(x) = \Big(\sum_{\Delta_N} r_\rho(t)\chi_\rho(x)\Big)\hat{f}(x) = \sum_{\Delta_N} r_\rho(x)\chi_\rho(x)\hat{f}(x)$$

$$= \sum_{\Delta_N} r_\rho(t)(S_\rho f)\hat{\ }(x) = \Big(\sum_{\Delta_N} r_\rho(t) S_\rho f\Big)\hat{\ }(x).$$

By Theorem 2, $\|\Sigma_{\Delta_N} r_\rho(t)S_\rho f\|_{p,w} < C\|f\|_{p,w}$. The C here depends on Δ, p and w, but not t or N. Applying (3.1), as in the method following (3.3), $\|(\Sigma_{\Delta_N}|S_\rho f|^2)^{1/2}\|_{p,w} < \|f\|_{p,w}$. Letting $N \to \infty$ and using the Monotone Convergence Theorem yields

$$\|\gamma(f)\|_{p,w} = \left\|\Big(\sum_{\Delta} |S_\rho f|^2\Big)^{1/2}\right\|_{p,w} < C\|f\|_{p,w}.$$

The proof of the opposite inequality is proved in the same manner as in Theorem 3.1. This completes the proof of Theorem 3.2.

Theorem 3.2 is the 1-dimensional version of inequality (1.4). We now proceed with the proof for general n.

From Theorem 3.2 we deduce the 1-dimensional inequality

$$\left\|\sum_{\Delta} r_\rho(t)S_\rho f\right\|_{p,w} < C\|f\|_{p,w}. \tag{3.5}$$

In fact, if we order $\Delta = \{\rho_i\}_{i=1}^\infty$, then for $f \in L_w^p(\mathbf{R}^1)$, $\{(\Sigma_1^N|S_{\rho_i}f|^2)^{1/2}\}_{N=1}^\infty$ is Cauchy in $L_w^p(\mathbf{R}^1)$. If $N > M$, using Theorem 3.2 again,

$$\left\|\sum_1^N r_{\rho_i}(t)S_{\rho_i}f - \sum_1^M r_{\rho_i}(t)S_{\rho_i}f\right\|_{p,w} = \left\|\sum_{M+1}^N r_{\rho_i}(t)S_{\rho_i}f\right\|_{p,w} < C\left\|\Big(\sum_{M+1}^N |S_{\rho_i}f|^2\Big)^{1/2}\right\|_{p,w},$$

which implies $\{\Sigma_1^N r_{\rho_i}(t)S_{\rho_i}f\}_{N=1}^\infty$ is Cauchy in $L_w^p(\mathbf{R}^1)$. (3.5) follows from this.

Define $T_t f(x) = \Sigma_\Delta\, r_\rho(t)S_\rho f(x)$. Let T_{t_i} be the operator above acting only on the x_i-variable, with the other variables fixed. By considering functions of the form $f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$, we see that

$$T_t f = T_{t_n}T_{t_{n-1}} \cdots T_{t_1} f. \tag{3.6}$$

If $f \in (L^2 \cap L_w^p)(\mathbf{R}^n)$, then for almost every fixed $(n-1)$-tuple $(x_2, \dots, x_n)$, $f(\cdot, x_2, \dots, x_n) \in (L^2 \cap L_w^p)(\mathbf{R}^1)$ and $w(\cdot, x_2, \dots, x_n) \in A_p(\mathbf{R}^1, \mathfrak{R}_1)$, with an A_p constant independent of $x_2, \dots, x_n$. Applying the 1-dimensional result (3.5) gives

$$\int_{\mathbf{R}^1} |T_{t_1}f(x_1, x_2, \dots, x_n)|^p w(x_1, x_2, \dots, x_n)\, dx_1$$

$$< C^p \int_{\mathbf{R}^1} |f(x_1, x_2, \dots, x_n)|^p w(x_1, x_2, \dots, x_n)\, dx_1.$$

Integrating in the other $n - 1$ variables, we have

$$\|T_{t_1}f\|_{p,w} \leq C\|f\|_{p,w}. \tag{3.7}$$

The C here depends on p, w and the lacunary constant $\alpha > 1$ for the sequence defining the decomposition for the real line related to the x_1-variable. Similarly, (3.7) holds with T_{t_1} replaced by T_{t_i}, $i = 2, \ldots, n$, with a constant depending on the decomposition related to the x_i-variable.

Using (3.6) and applying (3.7) successively in each variable, we obtain $\|T_t f\|_{p,w} \leq C\|f\|_{p,w}$, with a constant independent of t. Integrating in t and using the n-dimensional analog of (3.1), we get

$$\|\gamma(f)\|_{p,w} = \left\|\left(\sum_{\Delta} |S_\rho f|^2\right)^{1/2}\right\|_{p,w} \leq C\|f\|_{p,w}.$$

The proof is now completed as in the case of Theorem 3.1.

Let $\check{f}$ denote the inverse Fourier transform of f. We now prove Theorem 2.

Let $f \in \mathcal{S}(\mathbf{R}^1)$ and define g by $\hat{g}(x) = m(x)\hat{f}(x)$. Let Δ be the dyadic decomposition of $\mathbf{R}^1$. For $\rho \in \Delta$ and $\xi \in \rho$, set $\chi_{\rho,\xi}(x) = \chi(\{x\colon x \in \rho \text{ and } x < \xi\})$ and define $S_{\rho,\xi}f$ by $(S_{\rho,\xi}f)\hat{}(x) = \chi_{\rho,\xi}(x)\hat{f}(x)$. Then

$$\begin{aligned}(S_\rho g)(x) &= ((S_\rho q)\hat{}\,)\check{}(x) = \int_{\mathbf{R}^1} e^{2\pi i x\cdot\xi}(S_\rho g)\hat{}(\xi)\, d\xi \\ &= \int_{\mathbf{R}^1} e^{2\pi i x\cdot\xi}\chi_\rho(\xi)\hat{g}(\xi)\, d\xi = \int_{\mathbf{R}^1} e^{2\pi i x\cdot\xi}\chi_\rho(\xi)m(\xi)\hat{f}(\xi)\, d\xi \\ &= \int_\rho m(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi. \end{aligned} \tag{3.8}$$

Suppose now that $\rho = [2^k, 2^{k+1}]$ and set $F(\xi) = \int_{2^k}^{\xi} \hat{f}(t)e^{2\pi i x\cdot t}\, dt$. Then, $F'(\xi) = \hat{f}(\xi)e^{2\pi i x\cdot\xi}$ almost everywhere. Integrating by parts the right-hand side of (3.8) gives

$$\begin{aligned}(S_\rho g)(x) &= m(\xi)F(\xi)\big|_{2^k}^{2^{k+1}} - \int_\rho F(\xi)\, dm(\xi) \\ &= m(2^{k+1})((S_\rho f)\hat{}\,)\check{}(x) - \int_\rho\left(\int \chi_{\rho,\xi}(t)\hat{f}(t)e^{2\pi i x\cdot t}\, dt\right) dm(\xi) \\ &= m(2^{k+1})(S_\rho f)(x) - \int_\rho ((S_{\rho,\xi}f)\hat{}\,)\check{}(x)\, dm(\xi) \\ &= m(2^{k+1})(S_\rho f)(x) - \int_\rho (S_{\rho,\xi}f)(x)\, dm(\xi). \end{aligned} \tag{3.9}$$

Therefore, by the condition on m,

$$\begin{aligned}|(S_\rho g)(x)|^2 &\leq \left(\int_\rho |(S_{\rho,\xi}f)(x)|^2|dm(\xi)| + |(S_\rho f)(x)|^2|m(2^{k+1})|\right) \\ &\qquad \cdot \left(\int_\rho |dm(\xi)| + |m(2^{k+1})|\right) \\ &\leq 2B\left(\int_\rho |(S_{\rho,\xi}f)(x)|^2|dm(\xi)| + B|(S_\rho f)(x)|^2\right). \end{aligned}$$

From here, it follows that

$$\int \gamma(g)^p(x)w(x)\,dx = \int\Big(\sum_{\Delta}|(S_\rho g)(x)|^2\Big)^{p/2} w(x)\,dx$$

$$\leqslant (2B)^{p/2}\int\Big\{\sum_{\Delta}\Big(\int_\rho |(S_{\rho,\xi}f)(x)|^2|dm(\xi)| + B|(S_\rho f)(x)|^2\Big)\Big\}^{p/2} w(x)\,dx$$

$$\leqslant (2B)^{p/2}C^p\int\Big\{\sum_{\Delta}|(S_\rho f)(x)|^2\Big(\int_\rho |dm(\xi)| + B\Big)\Big\}^{p/2} w(x)\,dx$$

$$\leqslant (2BC)^p\int \gamma(f)^p(x)w(x)\,dx,$$

once we show that

$$\int\Big(\sum_{i=1}^{N}\int_{\rho_i}|(S_{\rho_i,\xi}f)(x)|^2|dm(\xi)|\Big)^{p/2} w(x)\,dx$$

$$< C\int\Big\{\sum_{i=1}^{N}|(S_{\rho_i}f)(x)|^2\Big(\int_{\rho_i}|dm(\xi)|\Big)\Big\}^{p/2} w(x)\,dx. \qquad (3.10)$$

To conclude $\|g\|_{p,w} < C\|\gamma(g)\|_{p,w}$ we need Lemma 3.7 and the version of Lemma 3.6 with Δ the dyadic decomposition of $\mathbf{R}^1$. The proof for this special case depends only on Theorems 4 and 3.1. Thus, proving (3.10) will complete the proof of Theorem 2 since we know by Theorem 3.1 that $\|\gamma(f)\|_{p,w}$ is equivalent to $\|f\|_{p,w}$ when Δ is the dyadic decomposition of $\mathbf{R}^1$.

Notice that for $\xi \in \rho$,

$$(S_{\rho,\xi}f)\hat{}(x) = \chi_{\rho,\xi}(x)\hat{f}(x) = \chi_{\rho,\xi}(x)\chi_\rho(x)\hat{f}(x) = (S_{\rho,\xi}(S_\rho f))\hat{}(x),$$

so that $S_{\rho,\xi}f$ is a partial sum of $S_\rho f$. Now, divide each ρ_i of (3.10) into m equal parts by partitions $\xi_j^i, j = 0, 1, \ldots, m, i = 1, \ldots, N$. By Theorem 2.1,

$$\int\Big\{\sum_{i=1}^{N}\Big(\sum_{j=1}^{m}|(S_{\rho_i,\xi_j^i}f)(x)|^2\int_{\xi_{j-1}^i}^{\xi_j^i}|dm(t)|\Big)\Big\}^{p/2} w(x)\,dx$$

$$< C^p\int\Big\{\sum_{i=1}^{N}\Big(\sum_{j=1}^{m}|(S_{\rho_i}f)(x)|^2\int_{\xi_{j-1}^i}^{\xi_j^i}|dm(t)|\Big)\Big\}^{p/2} w(x)\,dx$$

$$< C^p\int\Big\{\sum_{i=1}^{N}|(S_{\rho_i}f)(x)|^2\Big(\int_{\rho_i}|dm(t)|\Big)\Big\}^{p/2} w(x)\,dx.$$

Letting $m \to \infty$ proves (3.10) and thus Theorem 2.

We note that the proof of Theorem 3 is the same as Theorem 2. We decompose $S_\rho g$ into a sum of 2^n pieces each of which is handled as in Theorem 2.

The proof that inequality (1.4) implies $w \in A_p$ when Δ is the dyadic decomposition of $\mathbf{R}^n$ is contained in the following theorem. The result is true in the case where Δ is defined by the lacunary sequence $n_k = \alpha^k$ (or n lacunary decompositions $n_k^i = (\alpha_i)^k$); we consider only the dyadic case for simplicity. We follow the proof of Theorem 8 in [8].

THEOREM 3.3. *Let Δ be the dyadic decomposition of $\mathbf{R}^n$ and $1 < p < \infty$. If there exists a c such that*

$$w(\{x \in \mathbf{R}^n : |S_\rho f(x)| > \lambda\}) \leq (c/\lambda^p)\|f\|_{p,w}^p$$

for all $\rho \in \Delta$, then $w \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$.

Fix a rectangle $R = I_1 \times \cdots \times I_n$ and let f be positive on R and 0 elsewhere. Let k_j be the greatest integer such that $2^{k_j} \leq 1/4n|I_j|$ and let ρ be the dyadic rectangle $[2^{k_1}, 2^{k_1+1}] \times \cdots \times [2^{k_n}, 2^{k_n+1}]$. Note

$$\hat{\chi}_\rho(x) = \prod_{j=1}^n \hat{\chi}_{[2^{k_j}, 2^{k_j-1}]}(x_j) = \prod_{j=1}^n \left\{ \frac{\sin 2^{k_j-1}x_j}{x_j} e^{-i32^{k_j-1}x_j} \right\}$$

$$= \left\{ \prod_{j=1}^n 2^{k_j-1} \left[\frac{\sin 2^{k_j-1} x_j}{2^{k_j-1}x_j} \right] \right\} \exp\left(-i \sum_{j=1}^n 32^{k_j-1}x_j \right).$$

Since $S_\rho f(x) = (\hat{\chi}_\rho * f)(x)$, for $x \in R$ we have

$$|S_\rho f(x)| = \left| \int_R \frac{|\rho|}{2^n} \prod_{j=1}^n \left[\frac{\sin 2^{k_j-1}(x_j - y_j)}{2^{k_j-1}(x_j - y_j)} \right] \exp\left(-i \sum_{j=1}^n 32^{k_j-1}(x_j - y_j) \right) f(y)\, dy \right|$$

$$\geq \left| \operatorname{Re}\left\{ \int_R \frac{|\rho|}{2^n} \left[\prod_{j=1}^n \frac{\sin 2^{k_j-1}(x_j - y_j)}{2^{k_j-1}(x_j - y_j)} \right] \exp\left(-i \sum_{j=1}^n 32^{k_j-1}(x_j - y_j) \right) f(y)\, dy \right\} \right|$$

$$= \left| \int_R \frac{|\rho|}{2^n} \left[\prod_{j=1}^n \frac{\sin 2^{k_j-1}(x_j - y_j)}{2^{k_j-1}(x_j - y_j)} \right] \cos\left(\sum_{j=1}^n 32^{k_j-1}(x_j - y_j) \right) f(y)\, dy \right|. \quad (3.11)$$

By the definition of ρ and the fact that $x, y \in R$,

$$\left| \sum_{j=1}^n 32^{k_j-1}(x_j - y_j) \right| \leq \sum_{j=1}^n 22^{k_j}|I_j| \leq \sum_{j=1}^n \frac{1}{2n} = \frac{1}{2}.$$

Thus, $\cos(\sum_{j=1}^n 32^{k_j-1}(x_j - y_j)) \geq \cos(1/2) > 0$. Also,

$$|2^{k_j-1}(x_j - y_j)| \leq \frac{1}{2} \frac{1}{4n|I_j|} |I_j| \leq \frac{1}{8},$$

which implies that

$$\prod_{j=1}^n \frac{\sin 2^{k_j-1}(x_j - y_j)}{2^{k_j-1}(x_j - y_j)} \geq c > 0.$$

Therefore, for all $x \in R$, from (3.11) we get

$$|S_\rho f(x)| \geq C|\rho| \int_R f(y)\, dy \geq (C/|R|) \int_R f(y)\, dy,$$

since, by the definition of k_j,

$$\frac{1}{|R|} = \prod_{j=1}^{n} \frac{1}{|I_j|} < (4n)^n \prod_{j=1}^{n} 2^{k_j+1} = (8n)^n|\rho|.$$

Thus, by the assumed inequality,

$$\int_R w(x)\, dx \leqslant w\left(\left\{x \in \mathbf{R}^n : |S_\rho f(x)| > \frac{C}{|R|}\int_R f(y)\, dy\right\}\right)$$
$$\leqslant \frac{C}{\left(\frac{1}{|R|}\int f(y)\, dy\right)^p}\|f\|_{p,w}^p,$$

which can be rewritten

$$\left(\int_R w(x)\, dx\right)\left(\int_R f(y)\, dy\right)^p \leqslant C|R|^p \int_R |f(x)|^p w(x)\, dx. \tag{3.12}$$

To see this implies $w \in A_p$, we will show

$$\left(\frac{1}{|R|}\int_R w(x)\, dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\, dx\right)^{p-1} \leqslant C, \tag{3.13}$$

with the C in (3.12). Set $A = \int_R w(x)^{-1/(p-1)}\, dx$. If $A = 0$, the left-hand side of (3.13) is 0. If $0 < A < \infty$, set $f(x) = w(x)^{-1/(p-1)}$. The right-hand side of (3.12) equals $C|R|^pA$. Dividing both sides of (3.12) by $|R|^pA$ yields (3.13). If $A = \infty$, $w^{-1/p} \notin L^{p'}(R)$. Thus, there is a function $g \in L^p(R)$ such that $gw^{-1/p} \notin L^1(R)$. Let $f(x) = g(x)w(x)^{-1/p}$. From (3.12), we get $\int_R w(x)\, dx = 0$, so the left-hand side of (3.13) is 0.

To complete the proof of Theorem 1, we need to know that if $f \notin L_w^p(\mathbf{R}^n)$ then $\gamma(f) \notin L_w^p(\mathbf{R}^n)$, or equivalently, $\gamma(f) \in L_w^p(\mathbf{R}^n)$ implies $f \in L_w^p(\mathbf{R}^n)$. Since we need to be able to define the Fourier transform of f in order to form $\gamma(f)$, we may assume that f is at least in $\mathcal{S}'(\mathbf{R}^n)$, the space of tempered distributions. With this in mind, we will show that for $1 < p < \infty$ and $w \in A_p(\mathbf{R}^n, \mathfrak{D}_n)$, $\gamma(f) \in L_w^p(\mathbf{R}^n)$ implies $f \in L_w^p(\mathbf{R}^n)$ and $\|f\|_{p,w} \leqslant C\|\gamma(f)\|_{p,w}$. The proof will be a consequence of a few lemmas. Let $C_c^\infty(\mathbf{R}^n)$ be the space of C^∞ functions with compact support.

LEMMA 3.4. *Let* $\mathfrak{X} = \{\phi \in C_c^\infty(\mathbf{R}^n) : 0 \notin \operatorname{supp} \phi\}$. *Then* $\mathfrak{X}$ *is dense in* $L^2(\mathbf{R}^n)$.

Let $\beta \in C^\infty(\mathbf{R}^n)$ be such that $\beta(x) \equiv 1$ for $|x| > 1$ and $\beta(x) \equiv 0$ for $|x| < \frac{1}{2}$. Let $f \in L^2(\mathbf{R}^n)$ and $r_j \in C_c^\infty(\mathbf{R}^n)$ such that r_j converges to f in L^2. Set $\phi_j(x) = \beta(jx)r_j(x)$, so $\phi_j \in C_c^\infty(\mathbf{R}^n)$ and $\phi_j \equiv 0$ for $|x| < 1/(2j)$, hence $\phi_j \in \mathfrak{X}$. If $C = \sup_{x\in\mathbf{R}^n}|\beta(x)|$, then

$$\begin{aligned}\|f-\phi_j\|_2^2 &= \int_{\mathbf{R}^n} |f(x)-\phi_j(x)|^2\,dx\\ &\leq \int_{|x|>1/j} |f(x)-r_j(x)|^2\,dx + 2\int_{|x|<1/j} |f(x)|^2\,dx\\ &\quad +2\int_{|x|<1/j} |\beta(jx)r_j(x)|^2\,dx\\ &\leq \|f-r_j\|_2^2 + 2\int_{|x|<1/j}|f(x)|^2\,dx\\ &\quad +2C^2\left(\int_{|x|<1/j}|r_j(x)-f(x)|^2\,dx + \int_{|x|<1/j}|f(x)|^2\,dx\right)\\ &\leq (2C^2+2)\left\{\|f-r_j\|_2^2 + \int_{|x|<1/j}|f(x)|^2\,dx\right\}.\end{aligned}$$

Since $r_j \to f$ in L^2 and $\int_{|x|<1/j}|f(x)|^2\,dx \to 0$ as $j \to \infty$, $\|f-\phi_j\|_2 \to 0$.

From Lemma 3.4 we get

COROLLARY 3.5. *The set of $\phi \in \mathcal{S}(\mathbf{R}^n)$ such that $\check{\phi} \in \mathfrak{Z}$ is dense in $L^2(\mathbf{R}^n)$.*

Now, let $C = \{x \in \mathbf{R}^n\colon 1 \leq x_i \leq 2, i = 1, 2, \ldots, n\}$ and $\phi \in C_c^\infty(\mathbf{R}^n)$ be identically 1 on C and supported in $\{x \in \mathbf{R}^n\colon \frac{1}{2} < x_i < 4, i = 1, 2, \ldots, n\}$. For a dyadic interval, R_k, let ϕ_k be the function ϕ adjusted to R_k, as in proof of Theorem 3.1. The ϕ_k's have bounded overlaps; in fact, $1 \leq \sum \phi_k(x) \leq 2^n + 1$. Therefore, if we set $\psi_k = \phi_k/\sum \phi_k$, then $\psi_k \in C_c^\infty(\mathbf{R}^n)$ and $\sum \psi_k \equiv 1$. Let T_k be the operator with multiplier ψ_k.

LEMMA 3.6. *Let $1 < p < \infty$, $w \in A_p(\mathbf{R}^n, \mathfrak{D}_n)$, and $\gamma(f) \in L_w^p(\mathbf{R}^n)$. Then*

$$\left\|\sum T_k f\right\|_{p,w} \leq C\|\gamma(f)\|_{p,w}.$$

Since $\gamma(f) \in L_w^p(\mathbf{R}^n)$, $\sum_\Delta S_\rho f \in L_w^p(\mathbf{R}^n)$ and $\|\sum_\Delta S_\rho f\|_{p,w} \leq C\|\gamma(f)\|_{p,w}$. Let $T = \sum T_k$, so T is the operator with multiplier $\sum \psi_k = 1$. By Theorem 4, T is a bounded multiplier operator on $L_w^p(\mathbf{R}^n)$, so that

$$\left\|\sum T_k\left(\sum_\Delta S_\rho f\right)\right\|_{p,w} \leq C\left\|\sum_\Delta S_\rho f\right\|_{p,w} \leq C\|\gamma(f)\|_{p,w}.$$

The proof is complete once we know $T_k f = T_k(\sum_\Delta S_\rho f)$. Note that for M large enough, $\operatorname{supp}\psi_k \subset \bigcup_{\rho\in\Delta_M}\rho$ (Δ_M as in the proof of Theorem 3.2). Then, for all $N > M$,

$$(T_k f)\hat{} = \psi_k \hat{f} = \psi_k \sum_{\rho\in\Delta_N} \chi_\rho \hat{f} = \psi_k\left(\sum_{\rho\in\Delta_N} S_\rho f\right)\hat{} = \left(T_k\left(\sum_{\rho\in\Delta_N} S_\rho f\right)\right)\hat{},$$

so that $T_k f = T_k(\sum_{\rho\in\Delta_N}\chi_\rho f)$. But T_k is a bounded multiplier operator on $L_w^p(\mathbf{R}^n)$ and $\sum_{\rho\in\Delta_N} S_\rho f$ converges to $\sum_{\rho\in\Delta} S_\rho f$ in L_w^p. Therefore, since $T_k(\sum_{\rho\in\Delta_N} S_\rho f)$ is constant for $N > M$,

$$T_k\left(\sum_{\rho\in\Delta} S_\rho f\right) = T_k\left(\sum_{\rho\in\Delta_N} S_\rho f\right) = T_k f.$$

LEMMA 3.7. *Under the hypothesis of Lemma* 3.6, $\|f\|_{p,w} \leq \|\Sigma\, T_k f\|_{p,w}$.

Since $f \in \mathcal{S}'(\mathbf{R}^n)$, there are $g_j \in C_c^\infty(\mathbf{R}^n)$ such that g_j converges to f in $\mathcal{S}'$. Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ be such that $\check{\phi} \in \mathfrak{Z}$. Then

$$(g_j, \phi) = (\check{\hat{g}}_j, \phi) = (\hat{g}_j, \check{\phi}) = \int \hat{g}_j \check{\phi} = \int \Big(\sum_k \psi_k\Big)\hat{g}_j\check{\phi}$$

$$= \sum_k \int \hat{g}_j(\psi_k \check{\phi}) = \sum_k (\hat{g}_j, \psi_k\check{\phi}).$$

Because $\psi_k\check{\phi} \in \mathcal{S}(\mathbf{R}^n)$ and the Fourier transform is continuous on $\mathcal{S}'$, letting $j \to \infty$, we get

$$(f, \phi) = \sum_k (\hat{f}, \psi_k\check{\phi}) = \sum_k (\psi_k\hat{f}, \check{\phi}) = \sum_k ((T_kf)\hat{\ }, \check{\phi}) = \sum_k (T_kf, \phi).$$

This is actually a finite sum since $\check{\phi} \in \mathfrak{Z}$ implies $\psi_k\check{\phi} \equiv 0$ and consequently $(T_kf, \phi) = 0$ for almost every k. Therefore, $\Sigma(T_kf, \phi) = (\Sigma\, T_kf, \phi)$ and

$$|(f, \phi)| = \Big|\Big(\sum T_kf, \phi\Big)\Big| \leq \Big\|\sum T_kf\Big\|_{p,w}\|\phi\|_{p',w^{-1/(p-1)}}.$$

Since $\{\phi \in \mathcal{S}(\mathbf{R}^n): \check{\phi} \in \mathfrak{Z}\}$ is dense in L^2, taking the sup over such ϕ with $\|\phi\|_{p',w^{-1/(p-1)}} = 1$, we get $\|f\|_{p,w} \leq \|\Sigma\, T_kf\|_{p,w}$.

4. Applications. We now consider applications of Theorem 1. In particular, we will generalize Theorem 6 of Stein [**19**] and Theorems 3 and 4 of Riviere and Sagher [**16**].

Let $\{f_k\}$ be a sequence of functions defined on $\mathbf{R}^n$. By $\Sigma_k f_k \in L^p_w(\mathbf{R}^n)$ we mean the partial sums $\Sigma_1^N f_k$ converge in $L^p_w(\mathbf{R}^n)$.

THEOREM 4.1. *Let* $1 < p < \infty$, $w \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$, *and* $\{S_k\}$ *be any collection of lacunary partial sums. Then,* $(\Sigma_k|S_kf|^2)^{1/2} \in L^p_w(\mathbf{R}^n)$ *implies* $\Sigma_k\, \varepsilon_k S_kf \in L^p_w(\mathbf{R}^n)$ *for all* $\{\varepsilon_k\} \in l^\infty$. *Moreover, there is a constant, c, independent of f and* $\{\varepsilon_k\}$, *such that*

$$\Big\|\sum_k \varepsilon_k S_kf\Big\|_{p,w} \leq c\|\{\varepsilon_k\}\|_{l^\infty}\Big\|\Big(\sum_k |S_kf(x)|^2\Big)^{1/2}\Big\|_{p,w}.$$

Notice that since $(\Sigma_k|S_kf|^2)^{1/2} \in L^p_w(\mathbf{R}^n)$ and

$$\Big(\sum_k |\varepsilon_kS_kf(x)|^2\Big)^{1/2} \leq \|\varepsilon_k\|_{l^\infty}\Big(\sum |S_kf(x)|^2\Big)^{1/2},$$

$(\Sigma_k|\varepsilon_kS_kf|^2)^{1/2} \in L^p_w(\mathbf{R}^n)$. Thus, we may assume $\varepsilon_k = 1$ for all k. Using Theorem 1, the proof is the same as for inequality (3.5).

The converse of Theorem 4.1 is more general and easier to prove. We have

THEOREM 4.2. *Let* $p > 0$ *and* $w \geq 0$. *Let* $\{f_k\}$ *be any collection of functions and assume that* $\Sigma_k\, \varepsilon_kf_k \in L^p_w(\mathbf{R}^n)$ *for all* $\{\varepsilon_k\} \in l^\infty$. *Then* $(\Sigma_k|f_k|^2)^{1/2} \in L^p_w(\mathbf{R}^n)$ *and there exists a constant, c, independent of* $\{f_k\}$ *such that*

$$\Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_{p,w} \leq c \sup_{\|\{\varepsilon_k\}\|_{l^\infty}=1}\Big\|\sum_k \varepsilon_kf_k\Big\|_{p,w}.$$

It is enough to show that there exists a constant c, depending on $\{f_k\}$, such that

$$\left\| \sum_k \varepsilon_k f_k \right\|_{p,w} \leqslant c \| \{\varepsilon_k\} \|_{l^\infty} \tag{4.1}$$

for all $\{\varepsilon_k\} \in l^\infty$. For, if (4.1) is valid, it follows that $M = \sup_{\|\{\varepsilon_k\}\|_{l^\infty}=1} \|\sum_k \varepsilon_k f_k\|_{p,w}$ is finite. Let $\varepsilon_k = r_k(t)$ for $0 \leqslant t < 1$. Then $\|\{\varepsilon_k\}\|_{l^\infty} = 1$ and

$$M^p \geqslant \int_{\mathbf{R}^n} \left| \sum_k r_k(t) f_k(x) \right|^p w(x)\, dx.$$

Integrating in t, from 0 to 1, and using (3.1), we have

$$\begin{aligned} M^p &\geqslant \int_0^1 \int_{\mathbf{R}^n} \left| \sum_k r_k(t) f_k(x) \right|^p w(x)\, dx\, dt \\ &= \int_{\mathbf{R}^n} \left(\int_0^1 \left| \sum_k r_k(t) f_k(x) \right|^p dt \right) w(x)\, dx \\ &\geqslant c \int_{\mathbf{R}^n} \left(\sum_k |f_k(x)|^2 \right)^{p/2} w(x)\, dx. \end{aligned}$$

In order to prove (4.1), let $p \geqslant 1$ and consider the collection of maps $\{H_N: l^\infty \to L^p_w(\mathbf{R}^n)\}$ defined by $H_N(\{\varepsilon_k\}) = \sum_{k=1}^N \varepsilon_k f_k$. Let $H = H_\infty$. Since $H_N(\{\varepsilon_k\})$ is a finite sum, each H_N is continuous and by assumption $H_N(\{\varepsilon_k\})$ converges to $H(\{\varepsilon_k\})$ in $L^p_w(\mathbf{R}^n)$, for each $\{\varepsilon_k\} \in l^\infty$. Therefore, $\{\|H_N(\{\varepsilon_k\})\|_{p,w}\}_{N=1}^\infty$ is bounded for each $\{\varepsilon_k\} \in l^\infty$. By the Principle of Uniform Boundedness, there exists a constant, $c > 0$, such that $\|H_N\| \leqslant c$ for all N. It follows that $\|H\| \leqslant c$.

For $0 < p < 1$, the proof is the same, using the extension of the Principle of Uniform Boundedness to quasinormed spaces (see [23]).

Let $\{S_k\}$ be any collection of lacunary partial sums and set $f_k = S_k f$. Then, combining Theorems 1, 4.1, and 4.2, we obtain

THEOREM 4.3. *Let $1 < p < \infty$, $w \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$, and $\{S_k\}$ be any collection of lacunary partial sum operators. Then $f \in L^p_w(\mathbf{R}^n)$ if and only if $\sum_k \varepsilon_k S_k f$ converges in $L^p_w(\mathbf{R}^n)$ for any sequence $\{\varepsilon_k\} \in l^\infty$. Moreover, $\|f\|_{p,w}$ is equivalent to $\sup_{\|\{\varepsilon_k\}\|_{l^\infty}=1} \|\sum_k \varepsilon_k S_k f\|_{p,w}$.*

Let f be a measurable function and $\lambda(s) = m(\{x \in \mathbf{R}^n: |f(x)| > s\})$ be the distribution function of f (with respect to Lebesgue measure). We define the nonincreasing rearrangement f_r of f by $f_r(t) = \inf_{\lambda(s) \leqslant t} s$ for $t > 0$. Next, set

$$\|f\|^*_{p,q} = \left(\int_0^\infty \left[t^{1/p} f_r(t) \right]^q \frac{dt}{t} \right)^{1/q}$$

if $1 \leqslant p, q < \infty$, and

$$\|f\|^*_{p,q} = \sup_{t>0} t^{1/p} f_r(t)$$

if $1 < p < \infty$ and $q = \infty$. We then define the space $L^{p,q}$ as $\{f: \|f\|^*_{p,q} < \infty\}$. We note in passing that $L^{p,p} = L^p$ and for $q_2 < q_1$,

$$\|f\|^*_{p,q_1} \leqslant \|f\|^*_{p,q_2}. \tag{4.2}$$

For details, see [7].

Let $\Delta = \{\rho\}$ be a lacunary decomposition of $\mathbf{R}^n$. Define

$$\|f\|_{l^\infty(L^{r,\infty})} = \sup_{\rho \in \Delta} \|f\chi_\rho\|^*_{r,\infty}$$

and $l^\infty(L^{r,\infty}) = l^\infty(L^{r,\infty}, \Delta) = \{f\colon \|f\|_{l^\infty(L^{r,\infty})} < \infty\}$. The generalizations of the results of Riviere and Sagher [16] are the following two theorems.

THEOREM 4.4. *Let* $1 < p < \infty$ *and* $\Delta = \{\rho\}$ *be a lacunary decomposition of* $\mathbf{R}^n$.
(i) *If* $1 < p \leq 2$, $0 \leq \alpha < 1/p'$, *and* $f \in L^p_{|x|^{\alpha p}}(\mathbf{R}^n)$, *then*

$$\left(\sum_\Delta \|(S_\rho f)\hat{}(x)|x|^{-\alpha}\|^{*2}_{p',p}\right)^{1/2} \leq c\|f\|_{p,|x|^{\alpha p}};$$

(ii) *if* $2 \leq p < \infty$, $0 \leq \alpha < 1/p$, *and* $\Sigma_\Delta\|(S_\rho f)\hat{}(x)|x|^\alpha\|^{*2}_{p',p} < \infty$, *then*

$$\|f\|_{p,|x|^{-\alpha p}} \leq c\left(\sum_\Delta \|(S_\rho f)\hat{}(x)|x|^\alpha\|^{*2}_{p',p}\right)^{1/2}.$$

THEOREM 4.5. *Let* $1 < p \leq 2 \leq q < \infty$, $1/r = 1/p - 1/q$, $0 \leq \alpha < 1/q$, *and* $0 \leq \beta < 1/p'$. *Given a bounded function* $m(x)$, *let* Tf *be the multiplier operator defined by* $(Tf)\hat{}(x) = m(x)\hat{f}(x)$. *If* $m(x)|x|^{\alpha+\beta} \in l^\infty(L^{r,\infty})$, *then* T *is a bounded operator from* $L^p_{|x|^{\beta p}}(\mathbf{R}^n)$ *to* $L^q_{|x|^{-\alpha q}}(\mathbf{R}^n)$.

Two results are needed to prove Theorems 4.4 and 4.5. We will use Theorem 1 and the following versions of Pitt's Theorem (see [5], [14], and [20]):
(i) if $1 < p \leq 2$ and $0 \leq \alpha < 1/p'$, then

$$\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^{p'}|x|^{-\alpha p' n}\,dx\right)^{1/p'} \leq c(n,p,\alpha)\left(\int_{\mathbf{R}^n} |f(x)|^p|x|^{\alpha pn}\,dx\right)^{1/p}; \tag{4.3}$$

(ii) if $1 < p < \infty$, $0 \leq \alpha < 1/p'$, and $\lambda = 2/p + \alpha - 1 \geq 0$, then

$$\left(\int_{\mathbf{R}^n} |\hat{f}(x)|^{p}|x|^{-\lambda pn}\,dx\right)^{1/p} \leq c(n,p,\alpha)\left(\int_{\mathbf{R}^n} |f(x)|^p|x|^{\alpha pn}\,dx\right)^{1/p}. \tag{4.4}$$

These inequalities are also true with the roles of f and $\hat{f}$ reversed.

Riviere and Sagher were interested in the unweighted version of Theorem 4.4 in order to find a unified proof of Paley's Theorem and a generalization of the Hausdorff-Young Theorem, known as Kellog's Theorem:

$$\left(\sum_k\left(\sum_{n \in B_k} |\hat{f}(n)|^{p'}\right)^{2/p'}\right)^{1/2} \leq C_p\|f\|_p,$$

for $1 < p \leq 2$, where $\{B_k\}$ is the dyadic decomposition of the integers. When $p \geq 2$, the inequality sign is reversed.

Notice, first, that (4.4) is already a weighted version of Paley's Theorem–if $1 < p \leq 2$, say, we have

$$\int_{\mathbf{R}^n} |\hat{f}(x)|^p|x|^{(2-p)n}|x|^{-\alpha pn}\,dx \leq c\int_{\mathbf{R}^n} |f(x)|^p|x|^{\alpha pn}\,dx.$$

As for Kellog's Theorem, if we use (4.2) with (i) and (ii) of Theorem 4.4, we obtain

$$\Big(\sum_{\Delta} \|(S_\rho f)\hat{}(x)|x|^{-\alpha}\|_{p'}^2\Big)^{1/2} \le c\|f(x)|x|^{\alpha}\|_p, \qquad 1 < p \le 2, \tag{4.5}$$

and

$$\|f(x)|x|^{-\alpha}\|_p \le c\Big(\sum_{\Delta} \|(S_\rho f)\hat{}(x)|x|^{\alpha}\|_{p'}^2\Big)^{1/2}, \qquad 2 \le p < \infty. \tag{4.6}$$

Writing these at length, we see that (4.5) and (4.6) are weighted, n-dimensional analogs of Kellog's Theorem; e.g., (4.5) becomes

$$\left(\sum_{\rho\in\Delta}\left(\int_\rho |\hat{f}(x)|^{p'}|x|^{-\alpha p'}\,dx\right)^{2/p'}\right)^{1/2} \le c\|f\|_{p,|x|^{\alpha p}}.$$

In order to prove Theorem 4.4, fix a p and α satisfying the conditions of (i) and choose an r for which $1 < r < p$ and $\alpha < 1/r'$. By (4.3),

$$\|\hat{f}(x)|x|^{-\alpha}\|^*_{r',r} \le c\|f(x)|x|^{\alpha}\|^*_{r,r}$$

and

$$\|\hat{f}(x)|x|^{-\alpha}\|^*_{2,2} \le c\|f(x)|x|^{\alpha}\|^*_{2,2}.$$

By the interpolation theorem for $L^{p,q}$ spaces (see [7]), these imply

$$\|\hat{f}(x)|x|^{-\alpha}\|^*_{p',q} \le c\|f(x)|x|^{\alpha}\|_{p,q}, \tag{4.7}$$

for $1 \le q < \infty$.

Since $0 \le \alpha < 1/p'$, $0 \le \alpha p < p - 1$ so that $|x|^{\alpha p} \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$. Therefore, by (4.7), Minkowski's inequality and Theorem 1,

$$\begin{aligned}\Big(\sum_{\Delta} \|(S_\rho f)\hat{}(x)|x|^{-\alpha}\|^{*2}_{p',p}\Big)^{1/2} &\le c\Big(\sum_{\Delta} \|S_\rho f(x)|x|^{\alpha}\|^{*2}_{p',p}\Big)^{1/2} \\ &= c\Big(\sum_{\Delta} \|S_\rho f(x)|x|^{\alpha}\|_p^2\Big)^{1/2} \le c\left\|\Big(\sum_{\Delta} |S_\rho f(x)|^2\Big)^{1/2}|x|^{\alpha}\right\|_p \\ &= c\|\gamma(f)\|_{p,|x|^{\alpha p}} \le c\|f\|_{p,|x|^{\alpha p}}.\end{aligned}$$

To prove (ii), we proceed as before, only now we use the version of (4.3) with the roles of f and $\hat{f}$ interchanged, obtaining (note $p \ge 2$)

$$\|f(x)|x|^{-\alpha}\|^*_{p,q} \le c\|\hat{f}(x)|x|^{\alpha}\|^*_{p',q}, \qquad 1 \le q < \infty. \tag{4.8}$$

Now $0 \le \alpha < 1/p$, so that $-1 < -\alpha p \le 0$ and $|x|^{-\alpha p} \in A_p(\mathbf{R}^n, \mathfrak{R}_n)$. Using Theorem 1, Minkowski's inequality and (4.8),

$$\begin{aligned}\|f(x)|x|^{-\alpha}\|_p &\le c\left\|\Big(\sum_{\Delta} |S_\rho f(x)|^2\Big)^{1/2}|x|^{-\alpha}\right\|_p \le c\Big(\sum_{\Delta} \|S_\rho f(x)|x|^{-\alpha}\|_p^2\Big)^{1/2} \\ &\le c\Big(\sum_{\Delta} \|(S_\rho f)\hat{}(x)|x|^{\alpha}\|^{*2}_{p',p}\Big)^{1/2}.\end{aligned}$$

This completes the proof of Theorem 4.4.

Theorem 4.5 follows from Theorem 4.4 and Hölder's inequality for $L^{p,q}$ spaces.

$$\begin{aligned}
\|Tf\|_{q,|x|^{-\alpha q}} &= \|Tf(x)|x|^{-\alpha}\|_q = \|Tf(x)|x|^{-\alpha}\|^*_{q,q} \\
&\leq c\Big(\sum_\Delta \|(S_\rho(Tf))\hat{\ }(x)|x|^\alpha\|^{*2}_{q',q}\Big)^{1/2} \\
&= c\Big(\sum_\Delta \|\chi_\rho(x)m(x)\hat{f}(x)|x|^\alpha\|^{*2}_{q',q}\Big)^{1/2} \\
&= c\Big(\sum_\Delta \|\{m(x)|x|^{\alpha+\beta}\chi_\rho(x)\}\{\chi_\rho(x)\hat{f}(x)|x|^{-\beta}\}\|^{*2}_{q',q}\Big)^{1/2} \\
&\leq c\Big(\sum_\Delta \{\|m(x)|x|^{\alpha+\beta}\chi_\rho(x)\|^*_{r,\infty}\|\chi_\rho(x)\hat{f}(x)|x|^{-\beta}\|^*_{p',p}\}^2\Big)^{1/2} \\
&\leq c\|m(x)|x|^{\alpha+\beta}\|_{l^\infty(L^{r,\infty})}\Big(\sum_\Delta \|(S_\rho f)\hat{\ }(x)|x|^{-\beta}\|^{*2}_{p',p}\Big)^{1/2} \\
&\leq c\|m(x)|x|^{\alpha+\beta}\|_{l^\infty(L^{r,\infty})}\|f\|_{p,|x|^{\beta p}}.
\end{aligned}$$

The main steps in the previous proof are to apply a variant of Pitt's Theorem, Hölder's inequality for $L^{p,q}$ spaces, and another variant of Pitt's Theorem. If we use this procedure in the context of L^p spaces, we can prove

THEOREM 4.6. *Let* $1 < p, q < \infty$. *Given a bounded* $m(x)$, *define* $Tf(x)$ *by* $(Tf)\hat{\ }(x) = m(x)\hat{f}(x)$. *If*

(i) $1 < s \leq q, p \leq t < \infty$, *and* $1/r = 1/s - 1/t > 0$,

(ii) $\max(0, 1/s - 1/q') \leq \alpha < \min(1/q, 1/q + 1/s - 1/q')$,

(iii) $\max(0, 1/p' - 1/t) \leq \beta < \min(1/p', 1/t)$,

(iv) $m(x)|x|^{(\alpha+\beta+1/p+1/t-1/s-1/q)n} \in L^r(\mathbf{R}^n)$,

then T *is a bounded operator from* $L^p_{|x|^{\beta p}}(\mathbf{R}^n)$ *to* $L^q_{|x|^{-\alpha q}}(\mathbf{R}^n)$. *Moreover, if* $s = q > 2$, *we may take* $\alpha < 1/q$; *if* $t = p > 2$, *we may take* $\beta < 1/p'$.

Taking $s = q'$ and $t = p'$, we get Theorem 5. In 1-dimension, Theorem 4.5 is clearly better than Theorem 5 because $l^\infty(L^{r,\infty}) \supsetneqq L^r$. However, since Theorem 5 allows for a greater range of powers of $|x|$ for $n > 1$, in higher dimensions the two overlap. Finally, setting $p = q = s = t$, and noting the remark at the end of Theorem 4.6, we get

THEOREM 4.7. *Let* $1 < p < \infty$, $\max(0, (2-p)/p) \leq \alpha < 1/p$, *and*

$$\max(0, (p-2)/p) \leq \beta < 1/p'.$$

Let $m(x)$ *be bounded and* T *the multiplier operator defined by* m. *If* $m(x)|x|^{(\alpha+\beta)n} \in L^\infty(\mathbf{R}^n)$, *then* T *is a bounded operator from* $L^p_{|x|^{\beta p}}(\mathbf{R}^n)$ *to* $L^p_{|x|^{-\alpha p}}(\mathbf{R}^n)$.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907